

January 16, 1890.

Sir G. GABRIEL STOKES, Bart., President, in the Chair.

The Presents received were laid on the table, and thanks ordered for them.

The following Papers were read:—

- I. "On the Chief Line in the Spectrum of the Nebulæ." By J. NORMAN LOCKYER, F.R.S. Received December 9, 1889.

[Publication deferred.]

- II. "Observations regarding the Excretion and Uses of Bile." By A. W. MAYO ROBSON, F.R.C.S. Communicated by Dr. CLIFFORD ALLBUTT, F.R.S. Received December 24, 1889.

- III. "The Theory of Free Stream Lines." By J. H. MICHELL, Trin. Coll. Cam. Communicated by Professor J. J. THOMSON, F.R.S. Received January 3, 1890.

(Abstract.)

The chief object of the paper is to give a general method for the problem of free stream lines in two-dimensional motion of liquids when the boundaries are plane. The method rests on the transformation from one diagram to another by means of functions of complex variables, and, so far, is similar to that given by Kirchhoff in his 'Vorlesungen,' which is, however, of very limited application.

The first part is devoted to general theorems of transformation.

$$\begin{aligned} \text{If} \quad x + iy &\equiv z, & x - iy &\equiv z', \\ \phi + i\psi &\equiv \omega, & \phi - i\psi &\equiv \omega', \end{aligned}$$

and z is a function of ω , then the transformation theorems are obtained by considering the function

$$V \equiv \log \frac{dz}{d\omega} \cdot \frac{dz'}{d\omega'}.$$

V satisfies Laplace's equation. Let the intrinsic equation of the curve ψ_0 in the z plane be

$$s = f(k),$$

where s is the arc and k the curvature.

Then V satisfies the equation

$$\frac{V}{e^{\frac{V}{2}}} = \frac{d}{d\phi} f\left(\frac{d}{d\psi} e^{-\frac{V}{2}}\right)$$

along ψ_0 .

The transformation from one diagram to another is thereby reduced to solving a potential problem with a given condition over an infinite straight line.

If ψ_0 is a straight line, we have simply $dV/d\psi = 0$ along it.

By this means are given two general theorems, of which one is that of Schwarz, and the second may be stated as follows:—

The transformation

$$\frac{d\omega}{dz} = \Pi \frac{(z-z_r)^{m_r}}{(z-x_s)^{\frac{1}{2}}}$$

gives the potential of any number of infinitely long plane conductors, all in the same plane, with parallel edges and at given potentials.

Proceeding now to the hydrodynamical problem, x, y are the co-ordinates of a point of the liquid, ϕ, ψ , the velocity and stream functions.

The boundaries of the diagram in the ω plane are all straight, and therefore Schwarz's theorem will transform to a $u(\equiv p+iq)$ plane in which the boundary is $q = 0$.

Now, if v is the velocity of the fluid at (xy) ,

$$V \equiv \log \frac{dz}{d\omega} \cdot \frac{dz'}{d\omega'} = -\log v^2.$$

It follows that V is constant along part of the line $q = 0$ and $dV/dq = 0$ along the rest.

The transformation theorem just given enables us then to find V as a function of u , and therefore as a function of ω .

The general solution of the problem appears in the form

$$\frac{dz}{du} = \phi(u) e^{-\frac{i}{2} \int f(u) du}$$

where $\phi(u)$ and $f(u)$ are both factorial forms.

Several particular cases are next worked out, the results of which may here be given.

It will be observed that they are all of such a character that in going round the boundary of the z diagram we pass but twice from free stream line to rigid boundary.

It is only in these cases that the actual execution of the work is feasible.

Case I.—A Jet from a Vessel.

The general solution is

$$\frac{dz}{du} = \frac{1}{u-c} \Pi_n \left[\frac{1-a_n u + \sqrt{(1-a_n^2)} \sqrt{(1-u^2)}}{u-a_n} \right]^{1-a_n/\pi},$$

where

$$\frac{d\omega}{du} = \frac{1}{u-c}.$$

Ex. I.—A rectangular vessel of given width has an aperture in the bottom.

The elimination of the unknown constants is not possible in general, but if the aperture is symmetrically placed, and d be the breadth of the vessel, c of the aperture, and k of the jet, then

$$c = k \left[1 + \frac{1}{\pi} \left(\frac{d}{k} - \frac{k}{d} \right) \tan^{-1} \frac{2dk}{d^2 - k^2} \right].$$

Ex. II.—Tube projecting into the bottom of a vessel. The simplest results of this case are—

(a.) When the tube is very long

$$(d-c)^2 = d(d-k),$$

with the same notation as in Ex. I.

(b.) If the tube is of small length l , and the breadth of the vessel very large

$$k = c \frac{\pi}{\pi+2} \left(1 - \frac{4}{\pi+2} \sqrt{\frac{\pi+2}{\pi}} \frac{l}{c} \right).$$

Case II.—Flow from Pipes.

An aperture is made in one of the walls of a pipe along which water is flowing.

The formulæ of transformation are

$$\frac{dz}{du} = - \frac{au-1 + \sqrt{a^2-1} \sqrt{u^2-1}}{(u-b)(u-c)}$$

and

$$\frac{d\omega}{du} = - \frac{u-a}{(u-b)(u-c)}.$$

Let d be the breadth of the pipe,

k that of the aperture,

l of the jet,

v_1 the velocity in the pipe before reaching the aperture,

v_2 the velocity after the aperture is passed,

v_3 the velocity of the jet,

then

$$d(v_1 - v_2) = lv_3,$$

$$\text{and } \frac{k}{l} \pi = \frac{1}{2} \frac{v_1 + v_2}{v_3} \log \frac{(v_3 + v_1)^2}{(v_3 - v_1)^2} \frac{2v_3 - v_1 - v_2}{2v_3 + v_1 + v_2} \\ + \frac{v_3^2 + v_2^2}{v_3(v_1 - v_2)} \log \frac{(v_3 - v_2)(v_3 + v_1)}{(v_3 + v_2)(v_3 - v_1)} + \frac{\pi}{2} \left\{ \frac{4v_3^2 - (v_1 + v_2)^2}{v_3^2} \right\}^{\frac{1}{2}}.$$

If the water is flowing to the aperture equally from both sides, we get

$$k = l \left[1 + \frac{1}{\pi} \left(\frac{2d}{l} + \frac{l}{2d} \right) \log \frac{2d+l}{2d-l} \right].$$

If one end of the pipe is stopped, we have the case of a jet from the side of a vessel of width d , the aperture being far from the bottom.

In this case

$$\frac{k}{l} \pi = \left(\frac{d}{l} + \frac{l}{d} \right) \log \frac{d+l}{d-l} + \frac{1}{2} \frac{l}{d} \log \frac{2d-l}{2d+l} + \frac{\pi}{2} \left(4 - \frac{l^2}{d^2} \right)^{\frac{1}{2}}.$$

Lastly, if the pipe is very broad, so that we have a broad stream flowing past an aperture, the result is

$$\frac{k}{l} \pi = 2 \frac{v_3^2 + v_1^2}{v_3^2 - v_1^2} + \frac{v_1}{v_3} \log \frac{v_3 + v_1}{v_3 - v_1} + \pi \sqrt{\frac{v_3^2 - v_1^2}{v_3^2}},$$

v_3 being the velocity of the jet, and v_1 the velocity of the stream.

Case III.—Impact of a Stream against a Plane.

The stream impinges at a given angle against an infinite plane. If x be measured along the plane, the equations of the boundaries of the stream are

$$\left. \begin{aligned} x &= (1+a) \log \cos \frac{1}{2}\theta - (1-a) \log \sin \frac{1}{2}\theta - a \log \cos \theta \\ y &= \sqrt{(1-a^2)} \log \cot \frac{1}{2}(\frac{1}{2}\pi - \theta) + c \end{aligned} \right\}$$

and

$$\left. \begin{aligned} x &= (1+a) \log \sin \frac{1}{2}\theta - (1-a) \log \cos \frac{1}{2}\theta - a \log \cos \theta \\ y &= \sqrt{(1-a^2)} \log \cot \frac{1}{2}(\frac{1}{2}\pi - \theta) + c' \end{aligned} \right\},$$

where θ in both lies between 0 and $\frac{1}{2}\pi$, and $\cos^{-1} a$ is the inclination of the stream to the plane.

In the second part of the paper, some general transformation theorems are obtained, which are applicable to problems of electric condensers, forms of hollow vortices, &c.

If two polygons lie one within the other, the transformation of the area between them which makes the boundaries ψ curve is

$$\frac{dz}{dw} = \Pi_r \{ \Theta[a(w-w_r)] H[a(w-w_r)] \}^{\frac{\alpha_r}{\pi}-1},$$

where α_r is the internal angle of the polygon at $w = w_r$, and Θ, H are the elliptic functions usually so indicated.

A similar transformation is given for the case in which one polygon lies outside the other. The method is then applied to find the form of hollow vortices in certain cases. The transformation which gives the motion due to a stationary hollow vortex between two parallel planes is

$$z = A \log \tan (w - \tfrac{1}{2}iK').$$

Presents, January 16, 1890.

Transactions.

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